Proposition 1.20.1. *Given three arbitrary spaces E_1, E_2, E_3 there exists a linear isomorphism*

$$f: E_1 \otimes E_2 \otimes E_3 \xrightarrow{\cong} (E_1 \otimes E_2) \otimes E_3$$

such that

$$f(x \otimes y \otimes z) = (x \otimes y) \otimes z.$$

PROOF. Consider the trilinear mapping

$$E_1 \times E_2 \times E_3 \to (E_1 \otimes E_2) \otimes E_3$$

defined by

$$(x, y, z) \to (x \otimes y) \otimes z.$$

In view of the factorization property, there is induced a linear map

$$f: E_1 \otimes E_2 \otimes E_3 \to (E_1 \otimes E_2) \otimes E_3$$

such that

$$f(x \otimes y \otimes z) = (x \otimes y) \otimes z. \tag{1.14}$$

On the other hand, to each fixed $z \in E_3$ there corresponds a bilinear mapping $\beta_z: E_1 \times E_2 \to E_1 \otimes E_2 \otimes E_3$ defined by

$$\beta_z(x, y) = x \otimes y \otimes z.$$

The mapping β_z induces a linear map

$$g_z: E_1 \otimes E_2 \to E_1 \otimes E_2 \otimes E_3$$

such that

$$g_z(x \otimes y) = x \otimes y \otimes z. \tag{1.15}$$

Define a bilinear mapping

$$\psi: (E_1 \otimes E_2) \times E_3 \to E_1 \otimes E_2 \otimes E_3$$

by

$$\psi(u, z) = g_z(u) \qquad u \in E_1 \otimes E_2, z \in E_3. \tag{1.16}$$

Then ψ induces a linear map

$$g: (E_1 \otimes E_2) \otimes E_3 \to E_1 \otimes E_2 \otimes E_3$$

such that

$$\psi(u, z) = g(u \otimes z) \qquad u \in E_1 \otimes E_2, z \in E_3. \tag{1.17}$$

Combining (1.17), (1.16), and (1.15) we find

$$g((x \otimes y) \otimes z) = \psi(x \otimes y, z) = g_z(x \otimes y) = x \otimes y \otimes z. \tag{1.18}$$

Equations (1.14) and (1.18) yield $gf(x \otimes y \otimes z) = x \otimes y \otimes z$ and $fg((x \otimes y) \otimes z) = (x \otimes y) \otimes z$ showing that f is a linear isomorphism of $E_1 \otimes E_2 \otimes E_3$ onto $(E_1 \otimes E_2) \otimes E_3$ and g is the inverse isomorphism. □